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Surjections on Grassmannians preserving pairs of elements with bounded distance

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ABSTRACT

Let m and k be two fixed positive integers such that $m > k \geqslant 2$. Let V be a left vector space over a division ring with dimension at least $m + k + 1$. Let $\mathcal{G}_m(V)$ be the Grassmannian consisting of all m-dimensional subspaces of V. We characterize surjective mappings T from $\mathcal{G}_m(V)$ onto itself such that for any A, B in $\mathcal{G}_m(V)$, the distance between A and B is not greater than k if and only if the distance between $T(A)$ and $T(B)$ is not greater than k.

1. Introduction

Let V be a left vector space over a division ring D. For each positive integer m, let $\mathcal{G}_m(V)$ be the Grassmannian consisting of all m-dimensional subspaces of V. Two elements A, B of $\mathcal{G}_m(V)$ are at *distance* r, denoted by $d(A, B) = r$, if $\dim(A \cap B) = m - r$. They are called *adjacent* if $d(A, B) = 1$.

Suppose that $n = \dim V < \infty$ and $n - 1 > m > 1$. Then the well known Chow's theorem [2] states that if T is a bijection on $\mathcal{G}_m(V)$ that preserves adjacency in both directions, i.e.

$$d(A, B) = 1 \iff d(T(A), T(B)) = 1$$

for any A, B in $\mathcal{G}_m(V)$, then T is induced by a semi-linear isomorphism on V or when $n = 2m$, induced by a semi-linear isomorphism from V^* to V where V^* is the dual space of V. It is known that if T is a bijection on $\mathcal{G}_m(V)$ such that

$$d(A, B) \text{ is maximal } \iff d(T(A), T(B)) \text{ is maximal}$$

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for any A, B in $\mathcal{G}_m(V)$, then T preserves adjacency in both directions. This result was proved by Blunck and Havlicek [1] for the case $n = 2m$, and it was shown later [3] that the proof given in [1] works in the general case. Huang and Havlicek [6] proved a similar result for a general class of graphs that includes the above characterization as a special case. This class of graphs also include (i) graphs of rectangular matrices and (ii) graphs of certain hermitian matrices, both over division rings, where two matrices are adjacent if the rank of their difference is one.

In this note, we generalize the result of Blunck and Havlicek as follows: Let k be a fixed positive integer and $\dim V \geqslant m + k + 1$. If T is a surjection from $\mathcal{G}_m(V)$ onto itself such that

$$d(A, B) \leqslant k \iff d(T(A), T(B)) \leqslant k$$

for any A, B in $\mathcal{G}_m(V)$, then T is a bijection that preserves adjacency in both directions.

2. Preservers of pairs of subspaces with bounded distance

Throughout this section, m, k are two positive integers such that $m > k$, V and V' are left vector spaces over division rings D and D' respectively.

For characterizing surjective mappings from $\mathcal{G}_m(V)$ to $\mathcal{G}_{m'}(V')$ that preserve pairs of elements with distance bounded by k, we need the following concept.

For any non-empty subsets S of $\mathcal{G}_m(V)$, let

$$S^{\perp_k} = \{B \in \mathcal{G}_m(V) : d(A, B) \leqslant k \quad \forall A \in S\}.$$

For the following two lemmas, we assume that $\dim V \geqslant m + k + 1$.

Lemma 2.1. *Let $A, B \in \mathcal{G}_m(V)$ such that $0 < d(A, B) \leqslant k$. If $Z \in \{A, B\}^{\perp_k \perp_k}$, then $Z \supseteq A \cap B$.*

Proof. Let $M := A \cap B \cap Z$. We shall show that $\dim M \geqslant m - k$. Suppose the contrary that $\dim M =: t < m - k$. Choose a subspace J of $A \cap B$ of dimension $m - k - t$ such that $M \cap J = \{0\}$. Since $m \geqslant k + t + 1$, we are able to choose a $(k + 1)$-dimensional subspace H of Z such that $H \cap M = \{0\}$. Clearly $H \cap (M + J) = \{0\}$. Note that

$$\dim(H + M + J) = k + 1 + t + (m - k - t) = m + 1.$$

Since $\dim V \geqslant m + k + 1$, it follows that there exists a k-dimensional subspace U of V such that

$$U \cap (H + M + J) = \{0\}.$$

Note that

$$M + J + U \in \mathcal{G}_m(V).$$

Since

$$(M + J + U) \cap A \supseteq M + J,$$
$$(M + J + U) \cap B \supseteq M + J$$

and

$$\dim(M + J) = m - k,$$

it follows that

$$M + J + U \in \{A, B\}^{\perp_k}.$$

Since $(M + J + U) \cap H = \{0\}$ and the co-dimension of H in Z is $m - (k + 1)$, we have

$$\dim [(M + J + U) \cap Z] \leqslant m - k - 1,$$

a contradiction since $Z \in \{A, B\}^{\perp_k \perp_k}$. Hence $\dim M \geqslant m - k$.

Suppose that $Z \not\supseteq A \cap B$. Choose a subspace N of M of dimension $m - k - 1$. Let $u \in (A \cap B) \backslash Z$. Since $\dim(Z + \langle u \rangle) = m + 1$ and $\dim V \geqslant m + k + 1$, there exists a k-dimensional subspace W of V such that

$$W \cap (Z + \langle u \rangle) = \{0\}.$$

Let $X = \langle u \rangle + N + W$. Then $\dim X = m$ and

$$X \cap A \supseteq \langle u \rangle + N,$$
$$X \cap B \supseteq \langle u \rangle + N.$$

This implies that $X \in \{A, B\}^{\perp_k}$. However $Z \cap X = N$ and $\dim N = m - k - 1$. This implies that $Z \notin \{A, B\}^{\perp_k \perp_k}$, a contradiction. Hence $Z \supseteq A \cap B$. $\square$

If $d(A, B) = 1$, we call $\{M \in \mathcal{G}_m(V) : A \cap B \subseteq M \subseteq A + B\}$ the line joining A and B.

Lemma 2.2. *Let $A, B \in \mathcal{G}_m(V)$ such that $0 < d(A, B) \leqslant k$. Then*

(i) *$\{A, B\}^{\perp_k \perp_k}$ is the line joining A, B if $d(A, B) = 1$,*
(ii) *$\{A, B\}^{\perp_k \perp_k} = \{A, B\}$ if $d(A, B) \neq 1$.*

Proof. Let $Z \in \{A, B\}^{\perp_k \perp_k}$. By Lemma 2.1, we have $A \cap B \subseteq Z$. Let

$$x \in A \backslash (A \cap B) \text{ and } y \in B \backslash (A \cap B).$$

Then $\dim(Z + \langle x \rangle + \langle y \rangle) \leqslant m + 2$ and hence there exists a subspace W of V of dimension $k - 1$ such that

$$W \cap (Z + \langle x \rangle + \langle y \rangle) = \{0\}. \tag{2.1}$$

Let M be any subspace of $A \cap B$ of dimension $m - k - 1$. Then

$$N(x, y) := M + \langle x \rangle + \langle y \rangle + W \in \mathcal{G}_m(V),$$
$$N(x, y) \cap A \supseteq M + \langle x \rangle$$

and

$$N(x, y) \cap B \supseteq M + \langle y \rangle.$$

This shows that $N(x, y) \in \{A, B\}^{\perp_k}$. Hence

$$\dim (Z \cap N(x, y)) \geqslant m - k.$$

Since $\dim M = m - k - 1$, it follows that

$$w + \alpha x + \beta y \in Z \backslash \{0\}$$

for some $w \in W$ and some $\alpha, \beta \in D$. Let $z := w + \alpha x + \beta y$. Then

$$w = z - \alpha x - \beta y.$$

In view of (2.1), we have $w = 0$ and hence

$$Z \cap \langle x, y \rangle \neq \{0\}. \tag{2.2}$$

Case (i): $d(A, B) = 1$. We see that $Z \subseteq A + B$ and hence Z is in the line joining A and B. Now let $H \in \mathcal{G}_m(V)$ such that $A \cap B \subseteq H \subseteq A + B$, $H \neq A$ and $H \neq B$. Then $\lambda x + \mu y \in H$ for some nonzero scalars λ, μ. Let $J \in \{A, B\}^{\perp_k}$, $J \neq A$ and $J \neq B$. Then

$$\dim(J \cap A) \geqslant m - k \text{ and } \dim(J \cap B) \geqslant m - k. \tag{2.3}$$

Let $s = m - k$. Since $A \cap B$ is a hyperplane of A and $\dim(J \cap A) \geqslant s$, it follows that $\dim(A \cap B \cap J) \geqslant s - 1$. If $\dim(A \cap B \cap J) \geqslant s$, then $\dim(H \cap J) \geqslant s$. Now suppose that $\dim(A \cap B \cap J) = s - 1$. In view of (2.3) we have

$$u + \delta x \in J \quad \text{and} \quad v + \eta y \in J$$

for some non-zero scalars $\delta, \eta \in D$ and some $u, v \in A \cap B$. Now

$$\lambda\delta^{-1}(u + \delta x) + \mu\eta^{-1}(v + \eta y) \in J.$$

Hence

$$w + \lambda x + \mu y \in J \quad \text{for some } w \in A \cap B.$$

This proves that

$$J \cap H \supseteq J \cap A \cap B + \langle w + \lambda x + \mu y \rangle$$

and hence $\dim(H \cap J) \geqslant s$. This shows that $H \in \{A, B\}^{\perp_k \perp_k}$. Hence $\{A, B\}^{\perp_k \perp_k}$ is the line joining A and B.

Case (ii): $d(A, B) = j \neq 1$. We have $\dim(A \cap B) = m - j$. Let $x_1, x_2, \ldots, x_j$ be j linearly independent vectors in A and $y_1, y_2, \ldots, y_j$ be j linearly independent vectors in B such that

$$A + B = (A \cap B) \oplus \langle x_1, \ldots, x_j \rangle \oplus \langle y_1, \ldots, y_j \rangle.$$

Assume that $Z \neq A$ and $Z \neq B$. Without loss of generality, we may assume that $x_1 \notin Z$ and $y_1 \notin Z$. It follows from (2.2) that $\alpha_0 x_1 + \beta_0 y_2,\ \alpha_1 x_1 + \beta_1 y_1,\ \alpha_2 x_2 + \beta_2 y_1, \ldots, \alpha_j x_j + \beta_j y_j \in Z \backslash \{0\}$ for some $\alpha_0, \ldots, \alpha_j, \beta_0, \ldots, \beta_j \in D$. Since $\beta_0, \beta_1, \alpha_2$ are non-zero, we see that these $j + 1$ vectors are linearly independent and hence $\dim Z \geqslant (m - j) + j + 1 = m + 1$, a contradiction. Hence $Z \in \{A, B\}$. This shows that $\{A, B\}^{\perp_k \perp_k} = \{A, B\}$. $\square$

Theorem 2.3. *Let k and m be fixed positive integers such that $k < m$. Let V be a vector space of dimension at least $m + k + 1$. If T is a surjective map from $\mathcal{G}_m(V)$ to $\mathcal{G}_{m'}(V')$ such that*

$$d(A, B) \leqslant k \iff d(T(A), T(B)) \leqslant k$$

for any A, B in $\mathcal{G}_m(V)$, then T is bijective and T preserves adjacency in both directions.

Proof. By Lemma 2.2, it suffices to show that T is bijective. Suppose that $T(A) = T(B)$ for some distinct elements A, B. Then $d(T(A), T(B)) = 0 \leqslant k$. It follows that $d(A, B) \leqslant k$ and hence $\dim(A \cap B) = m - s$ where $0 < s \leqslant k$. Let N be an $(m - k - 1)$-dimensional subspace of $A \cap B$. Let M be an s-dimensional subspace of B such that

$$M \cap (A \cap B) = \{0\}.$$

Since $\dim V \geqslant m + k + 1$ and $\dim(A + B) = m + s$, it follows that there exists a $(k + 1 - s)$-dimensional subspace W of V such that $W \cap (A + B) = \{0\}$. Let

$$Z = W + N + M.$$

Then $\dim Z = m$. Note that

$$\dim(Z \cap A) = m - k - 1,$$
$$\dim(Z \cap B) \geqslant m - k.$$

Hence $d(Z, A) > k$ and $d(Z, B) \leqslant k$. By hypothesis,

$$d(T(Z), T(A)) > k \quad \text{and} \quad d(T(Z), T(B)) \leqslant k,$$

a contradiction since $T(A) = T(B)$. This shows that T is injective and the proof is complete. $\square$

Remark 2.4. If T is a bijective map from $\mathcal{G}_m(V)$ to $\mathcal{G}_{m'}(V')$ that preserves adjacency in both directions where $m, m' \geqslant 2$ and V has dimension at least $m + 2$, then $m = m'$, $\dim V = \dim V'$ and T is either induced by a semi-linear isomorphism from V to V' associated with an isomorphism from D to D' or when $\dim V = 2m$, induced by a semi-linear isomorphism from V^* to V' associated with an anti-isomorphism from D to D' where V^* is the dual space of V and is viewed as a right vector space over D. This can be shown by slightly modifying the proof of Chow's theorem [2,8]. For a finite dimensional version of this result, see also [9, Theorem 3.52].

Remark 2.5. Let F be a field with at least three elements and $M_{m,n}(F)$ be the vector space of all $m \times n$ matrices over F where $m, n \geqslant 2$. The distance $d(A, B)$ between A, B in $M_{m,n}(F)$ is the rank of $A - B$. By using Hua's fundamental theorem of the geometry of rectangular matrices [5], Havlicek and Šemrl [4] characterized all bijective maps T on $M_{m,n}(F)$ such that for all A, B in $M_{m,n}(F)$,

$$d(A, B) \text{ is maximal } \iff d(T(A), T(B)) \text{ is maximal.}$$

This result is closely connected with the theorem concerning bijections on Grassmannians preserving pairs of elements with maximal distance [1,3] and was extended by Lim and Tan [7] who characterized all surjective maps T on $M_{m,n}(F)$ such that for any A, B in $M_{m,n}(F)$,

$$d(A, B) \leqslant k \iff d(T(A), T(B)) \leqslant k$$

where k is a fixed integer such that $\min\{m, n\} > k \geqslant 2$.

The following was proved in [1,3,6] for finite dimensional vector spaces V and V'.

Corollary 2.6. *Let V be a vector space of dimension at least $m + 2$ where $m \geqslant 2$. If T is a surjective map from $\mathcal{G}_m(V)$ to $\mathcal{G}_{m'}(V')$ such that*

$$d(A, B) \textit{ is maximal } \iff d(T(A), T(B)) \textit{ is maximal}$$

for any A, B in $\mathcal{G}_m(V)$, then T is bijective and T preserves adjacency in both directions.

Proof. Let s denote the maximal distance between any two elements in $\mathcal{G}_m(V)$. Then $s = m$ if $\dim V \geqslant 2m$ and $s < m$ if $\dim V = m + s$. It follows from the hypothesis that

$$d(A, B) \leqslant s - 1 \iff d(T(A), T(B)) \leqslant s - 1$$

for any A, B in $\mathcal{G}_m(V)$. Hence the result follows from Theorem 2.3. □

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